

# Generalized almost paracontact structures

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## Abstract

The notion of generalized almost paracontact structure on the generalized tangent bundle  $TM \oplus T^*M$  is introduced and its properties are investigated. The case when the manifold  $M$  carries an almost paracontact metric structure is also discussed. Conditions for its transformed under a  $\beta$ - or a  $B$ -field transformation to be also a generalized almost paracontact structure are given. Finally, the normality of a generalized almost paracontact structure is defined and a characterization of a normal generalized almost paracontact structure induced by an almost paracontact one is given.

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**Key Words:** paracontact structure; generalized geometry.

## 1 Introduction

Generalized complex geometry unifies complex and symplectic geometry and proved to have applications in physics, for example, in quantum field theory, providing new sigma models [15]. N. Hitchin [7] initiated the study of generalized complex manifolds, continued by M. Gualtieri whose PhD thesis [5] is an outstanding paper on this subject. Afterwards, many authors investigated the geometry of the generalized tangent bundle from different points of view: M. Crainic [4] studied these structures from the point of view of Poisson and Dirac geometry, H. Bursztyn, G. R. Cavalcanti and M. Gualtieri [2], [3] presented a theory of reduction for generalized complex, generalized Kähler and hyper-Kähler structures. Regarding also the generalized Kähler manifolds, L. Ornea and R. Pantilie [10] discussed the integrability of the eigendistributions of the operator  $J_+J_- + J_-J_+$ , where  $J_{\pm}$  are the two almost Hermitian structures of a bihermitian one. In [9] they introduced the notion of holomorphic map in the context of generalized geometry. M. Abouzaid and M. Boyarchenko [1] proved that every generalized complex manifold admits a canonical Poisson structure. They also proved a local structure theorem and showed that in a neighborhood, a "first-order approximation" to the generalized complex structure is encoded in the data of a constant  $B$ -field and a complex Lie algebra. A technical description of the  $B$ -field was given by N. Hitchin [6] in terms of connections on gerbes. Extending the almost contact structures to the generalized tangent bundle, I. Vaisman [14] introduced

the generalized almost contact structure and established conditions for it to be normal. Y. S. Poon and A. Wade [11] described the particular cases coming from classical geometry, namely, when a contact structure, an almost cosymplectic and an almost contact one define a generalized almost contact structure. While the contact structures are in correspondence with complex structures, the paracontact structures are in correspondence with product structures. Therefore, would be natural to consider paracontact structures in the context of generalized geometry.

Our aim is to define on the generalized tangent bundle a *generalized paracontact structure* which naturally extends the previous ones. By means of certain orthogonal symmetries of  $TM \oplus T^*M$ , namely, the  $\beta$ - and  $B$ -transforms, in the particular case when the generalized paracontact structure comes from an almost paracontact one, we shall study its invariance under  $\beta$ - and  $B$ -field transformations, respectively, and also provide a necessary and sufficient condition for it to be normal (Proposition 3.5).

Also in [12] it is proved that such structures carry certain Lie bialgebroid/quasi-Lie algebroid structures.

## 2 Definitions and properties

The notion of almost paracontact structure was introduced by I. Sato. According to his definition [13], an *almost paracontact structure*  $(\varphi, \xi, \eta)$  on an odd-dimensional manifold  $M$  consists of a  $(1, 1)$ -tensor field  $\varphi$ , called the *structure endomorphism*, a vector field  $\xi$ , called the *characteristic vector field* and a 1-form  $\eta$ , called the *contact form*, which satisfy the following conditions:

1.  $\varphi^2 = I - \eta \otimes \xi$ ;
2.  $\eta(\xi) = 1$ .

Moreover, if  $g$  is a pseudo-Riemannian metric on  $M$  such that  $g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y)$ , for any  $X, Y \in \Gamma(TM)$ , we shall call  $(\varphi, \xi, \eta, g)$  *almost paracontact metric structure*. Notice that from the definition we deduce that  $\varphi\xi = 0$ ,  $\eta \circ \varphi = 0$ ,  $\eta = i_\xi g$ ,  $g(\xi, \xi) = 1$  and  $g(\varphi X, Y) = -g(X, \varphi Y)$ , for any  $X, Y \in \Gamma(TM)$ .

From the tangent bundle  $TM$  we shall pass to the generalized tangent bundle  $TM \oplus T^*M$ , whose sections are pairs of objects consisting of a vector field and a 1-form and we shall adopt the notation  $X + \alpha \in \Gamma(TM \oplus T^*M)$ . Let  $g_0(X + \alpha, Y + \gamma) := \frac{1}{2}(\alpha(Y) + \gamma(X))$ ,  $X + \alpha, Y + \gamma \in \Gamma(TM \oplus T^*M)$ , be the neutral metric on  $TM \oplus T^*M$  (of signature  $(n, n)$ , where  $n$  is the dimension of  $M$ ).

Extending this structure to the generalized tangent bundle  $TM \oplus T^*M$ , we give the following definition:

**Definition 2.1.** We say that  $(\Phi, \xi, \eta)$  is a generalized almost paracontact structure if  $\Phi$  is an endomorphism of the generalized tangent bundle  $TM \oplus T^*M$ ,  $\xi$  is a vector field and  $\eta$  is a 1-form on  $M$  such that

1.  $g_0(\Phi(X+\alpha), Y+\gamma) = -g_0(X+\alpha, \Phi(Y+\gamma))$ , for any  $X+\alpha, Y+\gamma \in \Gamma(TM \oplus T^*M)$ ;
2.  $\Phi^2 = \begin{pmatrix} I - \eta \otimes \xi & 0 \\ 0 & (I - \eta \otimes \xi)^* \end{pmatrix}$ ;
3.  $\Phi \begin{pmatrix} \eta \otimes \xi & 0 \\ 0 & (\eta \otimes \xi)^* \end{pmatrix} = 0$ ;
4.  $\|\xi + \eta\|_{g_0} = 1$ .

Taking into account the first relation in the definition, the representation of the structure  $\Phi$  by classical tensor fields is  $\Phi = \begin{pmatrix} \varphi & \beta \\ B & -\varphi^* \end{pmatrix}$ , where  $\varphi$  is an endomorphism of the tangent bundle  $TM$ ,  $\varphi^*$  its dual map defined by  $(\varphi^*\alpha)(X) := \alpha(\varphi X)$ ,  $\alpha \in \Gamma(T^*M)$ ,  $X \in \Gamma(TM)$ ,  $\beta$  a bivector field and  $B$  a 2-form on  $M$  (both of them skew-symmetric) and from the second relation we obtain the following conditions:

$$\begin{cases} \varphi^2 + \beta B = I - \eta \otimes \xi \\ B\beta + (\varphi^*)^2 = (I - \eta \otimes \xi)^* \\ \varphi\beta - \beta\varphi^* = 0 \\ B\varphi - \varphi^*B = 0 \end{cases}$$

which are equivalent to

$$\begin{cases} \varphi^2 = I - \eta \otimes \xi - \beta B \\ \beta(\alpha, \varphi^*\gamma) = \beta(\varphi^*\alpha, \gamma) \\ B(X, \varphi Y) = B(\varphi X, Y) \end{cases},$$

for any  $X + \alpha, Y + \gamma \in \Gamma(TM \oplus T^*M)$ .

Finally, the last two relations imply  $\beta(\eta, \cdot) = 0$ ,  $B(\xi, \cdot) = 0$ ,  $\varphi\xi = 0$ ,  $\eta \circ \varphi = 0$  and respectively,  $\eta(\xi) = 1$ . Remark that if  $(\varphi, \xi, \eta)$  is an almost paracontact structure, then  $(\Phi, \xi, \eta)$  is a generalized almost paracontact structure, where  $\Phi := \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{pmatrix}$ . Indeed,  $\Phi^2 := \begin{pmatrix} \varphi^2 & 0 \\ 0 & (\varphi^*)^2 \end{pmatrix}$  and for any  $\alpha \in \Gamma(T^*M)$  and  $X \in \Gamma(TM)$ :

$$\begin{aligned} [(\varphi^*)^2\alpha](X) &:= \varphi^*(\alpha(\varphi X)) := \alpha(\varphi^2 X) = \alpha(X - \eta(X) \cdot \xi) \\ &= \alpha(X) - \eta(X) \cdot \alpha(\xi) = [\alpha - \alpha(\xi) \cdot \eta](X). \end{aligned}$$

We obtain

$$\begin{aligned} \Phi^2(X + \alpha) &:= \varphi^2 X + (\varphi^*)^2 \alpha \\ &= X - \eta(X) \cdot \xi + \alpha - \alpha(\xi) \cdot \eta \\ &= (X + \alpha) - [\eta(X) \cdot \xi + \alpha(\xi) \cdot \eta] \\ &= I(X + \alpha) - F(X + \alpha), \end{aligned}$$

where  $F(X + \alpha) := \eta(X) \cdot \xi + \alpha(\xi) \cdot \eta$ . Then we can write  $F(X + \alpha) = JX + J^*\alpha$ , for  $JX := \eta(X) \cdot \xi = (\eta \otimes \xi)X$  and its dual map  $(J^*\alpha)(X) := \alpha(JX) = \alpha(\eta(X) \cdot \xi) = \alpha(\xi)\eta(X) = [\alpha(\xi) \cdot \eta](X)$ . Therefore,  $F = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}$  and

$$\Phi^2 = I - F = \begin{pmatrix} I - J & 0 \\ 0 & (I - J)^* \end{pmatrix} = \begin{pmatrix} I - \eta \otimes \xi & 0 \\ 0 & (I - \eta \otimes \xi)^* \end{pmatrix}.$$

The other relations from the definition are obvious.

### 3 On the generalized almost paracontact structure induced by an almost paracontact one

In what follows we shall consider the case when the generalized almost paracontact structure  $(\Phi, \xi, \eta)$  comes from an almost paracontact structure  $(\varphi, \xi, \eta)$ , namely,

$$\Phi := \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{pmatrix}.$$

In this case, we call  $(\Phi, \xi, \eta)$  the generalized almost paracontact structure induced by  $(\varphi, \xi, \eta)$ .

**Example 3.1.** Let  $(\varphi_1, \xi_1, \eta_1)$  and  $(\varphi_2, \xi_2, \eta_2)$  be two almost paracontact structures on  $M$  and for any  $t \in [0, \frac{\pi}{2}]$ , consider the one-parameter family  $(\varphi_t, \xi_t, \eta_t)_{t \in [0, \frac{\pi}{2}]}$  defined by  $\varphi_t := \cos t \cdot \varphi_1 + \sin t \cdot \varphi_2$ ,  $\xi_t := \cos t \cdot \xi_1 + \sin t \cdot \xi_2$ ,  $\eta_t := \cos t \cdot \eta_1 + \sin t \cdot \eta_2$ . Denote by  $\Phi_1 := \begin{pmatrix} \varphi_1 & 0 \\ 0 & -\varphi_1^* \end{pmatrix}$  and  $\Phi_2 := \begin{pmatrix} \varphi_2 & 0 \\ 0 & -\varphi_2^* \end{pmatrix}$  the endomorphisms of the corresponding generalized almost paracontact structures. If  $\eta_i(\xi_j) = \delta_{ij}$ ,  $\varphi_i \xi_j = 0$ ,  $i, j \in \{1, 2\}$  and  $\varphi_1 \varphi_2 + \varphi_2 \varphi_1 = -(\eta_1 \otimes \xi_2 + \eta_2 \otimes \xi_1)$ , then  $\Phi_t := \cos t \cdot \Phi_1 + \sin t \cdot \Phi_2$ ,  $t \in [0, \frac{\pi}{2}]$ , defines a generalized almost paracontact structure. Indeed, we get  $\varphi_1^* \varphi_2^* + \varphi_2^* \varphi_1^* = (\varphi_1 \varphi_2 + \varphi_2 \varphi_1)^*$  and from our conditions we obtain  $\Phi_t^2 = \begin{pmatrix} A_t & 0 \\ 0 & B_t \end{pmatrix} = \begin{pmatrix} I - \eta_t \otimes \xi_t & 0 \\ 0 & (I - \eta_t \otimes \xi_t)^* \end{pmatrix}$ , where  $A_t := \cos^2 t \cdot \varphi_1^2 + \sin^2 t \cdot \varphi_2^2 + \cos t \cdot \sin t \cdot (\varphi_1 \varphi_2 + \varphi_2 \varphi_1)$  and  $B_t := \cos^2 t \cdot (\varphi_1^*)^2 + \sin^2 t \cdot (\varphi_2^*)^2 + \cos t \cdot \sin t \cdot (\varphi_1^* \varphi_2^* + \varphi_2^* \varphi_1^*)$ .

#### 3.1 Compatibility with generalized Riemannian metrics

Let  $(\varphi, \eta, \xi, g)$  be an almost paracontact metric structure on  $M$  and consider on  $TM \oplus T^*M$  the generalized Riemannian metric  $\mathcal{G}_{\tilde{g}}$  induced by  $\tilde{g}$ , for  $\tilde{g}$  a Riemannian metric compatible with  $\varphi$  [ $\tilde{g}(\varphi X, Y) = -\tilde{g}(X, \varphi Y)$ , for any  $X, Y \in \Gamma(TM)$ ]. A natural question is if the endomorphism of the induced generalized almost paracontact structure  $(\Phi, \eta, \xi)$  is compatible with this metric. First, recall that a *generalized Riemannian metric*  $\mathcal{G}$  is a positive definite metric on the generalized tangent bundle  $TM \oplus T^*M$  such that

1.  $g_0(\mathcal{G}(X + \alpha), \mathcal{G}(Y + \gamma)) = g_0(X + \alpha, Y + \gamma)$ , for any  $X + \alpha, Y + \gamma \in \Gamma(TM \oplus T^*M)$ ;
2.  $\mathcal{G}^2 = I$ .

Representing  $\mathcal{G}$  as  $\mathcal{G} = \begin{pmatrix} \varphi & \sharp_{g_1} \\ \flat_{g_2} & \varphi^* \end{pmatrix}$ , where  $\varphi$  is an endomorphism of the tangent bundle  $TM$ ,  $\varphi^*$  its dual map,  $\flat_{g_i}(X) := i_X g_i$ ,  $X \in \Gamma(TM)$  and  $\sharp_{g_i} := \flat_{g_i}^{-1}$ ,  $i \in \{1, 2\}$ , for  $g_1, g_2$  Riemannian metrics on  $M$ , the two conditions are equivalent to:

$$\begin{cases} \varphi^2 = I - \sharp_{g_1} \circ \flat_{g_2} \\ g_i(X, \varphi Y) = -g_i(\varphi X, Y) \end{cases} \quad ,$$

for any  $X, Y \in \Gamma(TM)$ ,  $i \in \{1, 2\}$ .

Let  $\tilde{g}$  be a Riemannian metric on  $M$  and consider the positive definite generalized metric  $\mathcal{G}_{\tilde{g}}$  [8], which can be viewed as an automorphism of  $TM \oplus T^*M$ ,  $\mathcal{G}_{\tilde{g}} := \begin{pmatrix} 0 & \sharp_{\tilde{g}} \\ \flat_{\tilde{g}} & 0 \end{pmatrix}$ , where  $\sharp_{\tilde{g}}$  is the inverse of the musical isomorphism  $\flat_{\tilde{g}}(X) := i_X \tilde{g}$ ,  $X \in \Gamma(TM)$ .

**Proposition 3.1.** *If  $(\varphi, \eta, \xi, g)$  is an almost paracontact metric structure on  $M$  and  $\tilde{g}$  is a Riemannian metric satisfying  $\tilde{g}(\varphi X, Y) = -\tilde{g}(X, \varphi Y)$ , for any  $X, Y \in \Gamma(TM)$ , then the endomorphism  $\Phi$  of the induced generalized paracontact structure is compatible with the generalized Riemannian metric  $\mathcal{G}_{\tilde{g}}$ , that is,  $\mathcal{G}_{\tilde{g}} \circ \Phi = -\Phi \circ \mathcal{G}_{\tilde{g}}$ .*

*Proof.* For any  $X + \alpha \in \Gamma(TM \oplus T^*M)$ ,  $\mathcal{G}_{\tilde{g}}(\Phi(X + \alpha)) := \mathcal{G}_{\tilde{g}}(\varphi X - \varphi^* \alpha) := \sharp_{\tilde{g}}(\varphi^* \alpha) - \flat_{\tilde{g}}(\varphi X)$ . Therefore, for any  $U \in \Gamma(TM)$ ,  $\tilde{g}(\sharp_{\tilde{g}}(\varphi^* \alpha), U) = \alpha(\varphi U)$  and  $(\flat_{\tilde{g}}(\varphi X))(U) = \tilde{g}(\varphi X, U)$ . But for any  $X + \alpha \in \Gamma(TM \oplus T^*M)$ ,  $\mathcal{G}_{\tilde{g}}(X + \alpha) := \sharp_{\tilde{g}}(\alpha) + \flat_{\tilde{g}}(X)$  and so, for any  $U \in \Gamma(TM)$ ,  $\tilde{g}(\sharp_{\tilde{g}}(\alpha), U) = \alpha(U)$  and  $(\flat_{\tilde{g}}(X))(U) = \tilde{g}(X, U)$ .

It follows

$$\tilde{g}(\sharp_{\tilde{g}}(\varphi^* \alpha), U) = \alpha(\varphi U) = \tilde{g}(\sharp_{\tilde{g}}(\alpha), \varphi U) = -\tilde{g}(\varphi(\sharp_{\tilde{g}}(\alpha)), U),$$

for any  $U \in \Gamma(TM)$  and so  $\sharp_{\tilde{g}}(\varphi^* \alpha) = -\varphi(\sharp_{\tilde{g}}(\alpha))$ .

Also

$$(\flat_{\tilde{g}}(\varphi X))(U) = \tilde{g}(\varphi X, U) = -\tilde{g}(X, \varphi U) = -(\flat_{\tilde{g}}(X))(\varphi U) = -(\varphi^*(\flat_{\tilde{g}}(X)))(U),$$

for any  $U \in \Gamma(TM)$  and so  $\flat_{\tilde{g}}(\varphi X) = -\varphi^*(\flat_{\tilde{g}}(X))$ .

Then, for any  $X + \alpha \in \Gamma(TM \oplus T^*M)$ :

$$\begin{aligned} \mathcal{G}_{\tilde{g}}(\Phi(X + \alpha)) &:= \sharp_{\tilde{g}}(\varphi^* \alpha) - \flat_{\tilde{g}}(\varphi X) \\ &= -(\varphi(\sharp_{\tilde{g}}(\alpha)) - \varphi^*(\flat_{\tilde{g}}(X))) \\ &:= -\Phi(\sharp_{\tilde{g}}(\alpha) + \flat_{\tilde{g}}(X)) \\ &= -\Phi(\mathcal{G}_{\tilde{g}}(X + \alpha)). \end{aligned}$$

□

**Remark 3.1.** From the previous computations we also deduce that  $\flat_{\tilde{g}} \circ \varphi = -\varphi^* \circ \flat_{\tilde{g}}$  (respectively,  $\sharp_{\tilde{g}} \circ \varphi^* = -\varphi \circ \sharp_{\tilde{g}}$ ).

### 3.2 Invariance under a $B$ -field transformation

Besides the diffeomorphisms, the Courant bracket (which extends the Lie bracket to the generalized tangent bundle) admits some other symmetries, namely, the  $B$ -field transformations. Now we are interested in what happens if we apply to the endomorphism  $\Phi$  a  $B$ -field transformation.

Let  $B$  be a closed 2-form on  $M$  [viewed as a map  $B : \Gamma(TM) \rightarrow \Gamma(T^*M)$ ] and consider the  $B$ -transform,  $e^B := \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}$ . We can define  $\Phi_B := e^B \Phi e^{-B}$  which has the expression  $\Phi_B = \begin{pmatrix} \varphi & 0 \\ B\varphi + \varphi^*B & -\varphi^* \end{pmatrix}$  and for any  $X + \alpha \in \Gamma(TM \oplus T^*M)$ , we have

$$\Phi_B(X + \alpha) = \varphi X + B(\varphi X) + \varphi^*(B(X)) - \varphi^*\alpha.$$

For any  $Y \in \Gamma(TM)$ , we get

$$[B(\varphi X) + \varphi^*(B(X)) - \varphi^*\alpha](Y) = B(\varphi X, Y) + B(X, \varphi Y) - (\varphi^*\alpha)(Y).$$

Note that if the 2-form  $B$  satisfies  $B(\varphi X, Y) = -B(X, \varphi Y)$ , for any  $X, Y \in \Gamma(TM)$ , then  $\Phi_B$  coincides with  $\Phi$ . In particular, if  $(\varphi, \eta, \xi, g)$  is an almost para-cosymplectic metric structure and if we take  $B(X, Y) := g(\varphi X, Y)$ ,  $X, Y \in \Gamma(TM)$ , we obtain

$$B(\varphi X, Y) := g(\varphi^2 X, Y) = -g(\varphi X, \varphi Y) := -B(X, \varphi Y)$$

and  $\Phi_B$  is just  $\Phi$ .

A sufficient condition on  $B$  for  $\Phi_B$  to define a generalized almost paracontact structure is given by the following proposition:

**Proposition 3.2.** *If the 2-form  $B$  satisfies  $B(\varphi^2 X, Y) = B(\varphi X, \varphi Y)$ , for any  $X, Y \in \Gamma(TM)$ , then  $(\Phi_B, \eta, \xi)$  is a generalized almost paracontact structure.*

*Proof.* Indeed,  $\Phi_B^2 = \begin{pmatrix} \varphi^2 & 0 \\ B\varphi^2 - (\varphi^*)^2 B & (\varphi^*)^2 \end{pmatrix} = \begin{pmatrix} I - \eta \otimes \xi & 0 \\ 0 & (I - \eta \otimes \xi)^* \end{pmatrix}$ .  $\square$

**Remark 3.2.** In the general case, if  $\Phi$  is represented  $\Phi = \begin{pmatrix} \varphi & \beta \\ B & -\varphi^* \end{pmatrix}$ , then its  $B$ -transform,  $\Phi_B = \begin{pmatrix} \varphi - \beta B & \beta \\ B\varphi + \varphi^*B + B - B\beta B & -\varphi^* + B\beta \end{pmatrix}$  defines a generalized almost paracontact structure.

### 3.3 Invariance under a $\beta$ -field transformation

Similarly we shall see what happens if we apply to the endomorphism  $\Phi$  a  $\beta$ -field transformation. Let  $\beta$  be a bivector field on  $M$  [viewed as a map  $\beta : \Gamma(T^*M) \rightarrow \Gamma(TM)$ ] and

consider the  $\beta$ -transform,  $e^\beta := \begin{pmatrix} I & \beta \\ 0 & I \end{pmatrix}$ . We can define  $\Phi_\beta := e^\beta \Phi e^{-\beta}$  which has the expression  $\Phi_\beta = \begin{pmatrix} \varphi & -\varphi\beta - \beta\varphi^* \\ 0 & -\varphi^* \end{pmatrix}$  and for any  $X + \alpha \in \Gamma(TM \oplus T^*M)$ , we have

$$\Phi_\beta(X + \alpha) = \varphi X - \varphi(\beta(\alpha)) - \beta(\varphi^*\alpha) - \varphi^*\alpha.$$

If the bivector field  $\beta$  satisfies  $\beta \circ \varphi^* = -\varphi \circ \beta$ , then  $\Phi_\beta$  coincides with  $\Phi$ .

A sufficient condition on  $\beta$  for  $\Phi_\beta$  to define a generalized almost paracontact structure is given by the following proposition:

**Proposition 3.3.** *If the bivector field  $\beta$  satisfies  $\eta(\beta(\alpha)) \cdot \xi = \alpha(\xi) \cdot \beta(\eta)$ , for any  $\alpha \in \Gamma(T^*M)$ , then  $(\Phi_\beta, \eta, \xi)$  is a generalized almost paracontact structure.*

*Proof.* Indeed,  $\Phi_\beta^2 = \begin{pmatrix} \varphi^2 & \beta(\varphi^*)^2 - \varphi^2\beta \\ 0 & (\varphi^*)^2 \end{pmatrix}$  and for any  $\alpha \in \Gamma(T^*M)$ :

$$\beta((\varphi^*)^2\alpha) - \varphi^2(\beta(\alpha)) = \beta(\alpha - \alpha(\xi) \cdot \eta) - (\beta(\alpha) - \eta(\beta(\alpha)) \cdot \xi) = \eta(\beta(\alpha)) \cdot \xi - \alpha(\xi) \cdot \beta(\eta) = 0$$

$$\text{and so } \Phi_\beta^2 = \begin{pmatrix} I - \eta \otimes \xi & 0 \\ 0 & (I - \eta \otimes \xi)^* \end{pmatrix}. \quad \square$$

**Remark 3.3.** In the general case, if  $\Phi$  is represented  $\Phi = \begin{pmatrix} \varphi & \beta \\ B & -\varphi^* \end{pmatrix}$ , then its  $\beta$ -transform,  $\Phi_\beta = \begin{pmatrix} \varphi + \beta B & -\varphi\beta - \beta\varphi^* + \beta - \beta B\varphi \\ B & -\varphi^* - B\beta \end{pmatrix}$  defines a generalized almost paracontact structure.

### 3.4 Paracontactomorphisms

We shall prove that a diffeomorphism between two almost paracontact manifolds preserving the almost paracontact structure induces a diffeomorphism between their generalized tangent bundles which preserves the generalized almost paracontact structure.

Let  $(M_1, \varphi_1, \xi_1, \eta_1)$  and  $(M_2, \varphi_2, \xi_2, \eta_2)$  be two almost paracontact manifolds.

**Definition 3.1.** We say that  $f : (M_1, \varphi_1, \xi_1, \eta_1) \rightarrow (M_2, \varphi_2, \xi_2, \eta_2)$  is a paracontactomorphism if  $f$  is a diffeomorphism and satisfies

$$\varphi_2 \circ f_* = f_* \circ \varphi_1, \quad f_* \xi_1 = \xi_2.$$

Remark that in this case,  $f^*\eta_2 = \eta_1$  is also implied. Indeed, for any  $X \in \Gamma(TM_1)$ , applying  $f_*$  to  $\varphi_1^2 X = X - \eta_1(X) \cdot \xi_1$ , we get

$$\begin{aligned} f_* X - (f^*)^{-1}(\eta_1(X)) \cdot f_* \xi_1 &= (f_* \circ \varphi_1)(\varphi_1 X) = \varphi_2((f_* \circ \varphi_1)X) \\ &= \varphi_2^2(f_* X) = f_* X - \eta_2(f_* X) \cdot \xi_2 \end{aligned}$$

and so,  $\eta_2(f_* X) \circ f = \eta_1(X)$ , for any  $X \in \Gamma(TM_1)$ .

**Lemma 3.1.** *If  $f : (M_1, \varphi_1, \xi_1, \eta_1) \rightarrow (M_2, \varphi_2, \xi_2, \eta_2)$  is a paracontactomorphism, then  $\varphi_1^* \circ f^* = f^* \circ \varphi_2^*$ .*

*Proof.* For any  $X \in \Gamma(TM_1)$ ,  $\alpha \in \Gamma(T^*M_2)$ ,  $x \in M_1$ :

$$[[(\varphi_1^* \circ f^*)(\alpha)](X)](x) = [(f^*\alpha)(\varphi_1 X)](x) = \alpha_{f(x)}(f_{*x}((\varphi_1 X)_x)) = [\alpha(f_*(\varphi_1 X))](f(x))$$

and respectively,

$$\begin{aligned} [[(f^* \circ \varphi_2^*)(\alpha)](X)](x) &= (\varphi_2^*\alpha)_{f(x)}(f_{*x}(X_x)) = [(\varphi_2^*\alpha)(f_*X)](f(x)) \\ &:= [\alpha(\varphi_2(f_*X))](f(x)) = [\alpha(f_*(\varphi_1 X))](f(x)). \end{aligned}$$

□

**Proposition 3.4.** *Let  $f : (M_1, \varphi_1, \xi_1, \eta_1) \rightarrow (M_2, \varphi_2, \xi_2, \eta_2)$  be a paracontactomorphism. Then it induces a diffeomorphism between their generalized tangent bundles,  $\tilde{f}(X + \alpha) := f_*X + (f^{-1})^*\alpha$ ,  $X + \alpha \in \Gamma(TM_1 \oplus T^*M_1)$ , such that  $\Phi_2 \circ \tilde{f} = \tilde{f} \circ \Phi_1$  and  $\tilde{f}(\xi_1 + 0) = \xi_2 + 0$ .*

*Proof.* Using the previous lemma, we obtain, for any  $X + \alpha \in \Gamma(TM_1 \oplus T^*M_1)$ :

$$\begin{aligned} (\Phi_2 \circ \tilde{f})(X + \alpha) &:= \Phi_2(f_*X + (f^{-1})^*\alpha) := (\varphi_2 \circ f_*)(X) + (\varphi_2^* \circ (f^{-1})^*)(\alpha) \\ &= (f_* \circ \varphi_1)(X) + ((f^{-1})^* \circ \varphi_1^*)(\alpha) \\ &:= \tilde{f}(\varphi_1 X + \varphi_1^* \alpha) := (\tilde{f} \circ \Phi_1)(X + \alpha). \end{aligned}$$

$$\text{Also, } \tilde{f}(\xi_1 + 0) = f_*\xi_1 + 0 = \xi_2 + 0 \text{ and } \tilde{f}(0 + \eta_1) = 0 + (f^{-1})^*\eta_1 = 0 + \eta_2. \quad \square$$

### 3.5 Normality of $(\Phi, \xi, \eta)$

I. Vaisman [14] defined normal generalized contact structures and characterized them. We give an analogue definition for the normality of a generalized almost paracontact structure like in the generalized contact case:

**Definition 3.2.** A generalized almost paracontact structure is called normal if the  $M$ -adapted generalized almost product structure on  $M \times \mathbb{R}$  is integrable.

Precisely, in our particular case, if  $(\Phi := \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{pmatrix}, \xi, \eta)$  is the generalized almost paracontact structure induced by the almost paracontact one  $(\varphi, \xi, \eta)$ , then the  $M$ -adapted generalized almost product structure is  $P = \begin{pmatrix} \varphi & \beta \\ B & -\varphi^* \end{pmatrix}$ , where  $\varphi^2 = I - \beta B$ ,  $\beta(\alpha, \varphi^* \gamma) = \beta(\varphi^* \alpha, \gamma)$ ,  $B(X, \varphi Y) = B(\varphi X, Y)$ , for any  $X, Y \in \Gamma(TM)$  and  $\alpha, \gamma \in \Gamma(T^*M)$ . Moreover, from the condition to be  $M$ -adapted [14] follows  $\beta = \xi \wedge \frac{\partial}{\partial t}$  and  $B = \eta \wedge dt$ . The integrability of  $P$  means that its Courant-Nijenhuis tensor field

$$\begin{aligned} \mathcal{N}_P(X + \alpha, Y + \gamma) &:= [P(X + \alpha), P(Y + \gamma)] + P^2[X + \alpha, Y + \gamma] \\ &\quad - P[P(X + \alpha), Y + \gamma] - P[X + \alpha, P(Y + \gamma)], \end{aligned}$$



for  $X + \alpha, Y + \gamma \in \Gamma(TM \oplus T^*M)$ , vanishes identically, where the Courant bracket is given by

$$[X + \alpha, Y + \gamma] := [X, Y] + L_X \gamma - L_Y \alpha + \frac{1}{2} d(\alpha(Y) - \gamma(X)).$$

Computing it we obtain the normality condition for  $(\Phi, \xi, \eta)$ :

$$\begin{cases} N_\varphi(X, Y) - d\eta(X, Y) \cdot \xi = 0 \\ L_\xi \eta = 0, L_\xi \varphi = 0 \\ (L_{\varphi X} \eta)Y - (L_{\varphi Y} \eta)X = 0 \end{cases},$$

for any  $X, Y \in \Gamma(TM)$ , where

$$N_\varphi(X, Y) := [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

**Proposition 3.5.** *The generalized almost paracontact structure  $(\Phi := \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{pmatrix}, \xi, \eta)$  induced by the almost paracontact one  $(\varphi, \xi, \eta)$  is normal if and only if  $(\varphi, \xi, \eta)$  is normal.*

*Proof.* The first implication is trivial. For the converse one, it is known that  $(\varphi, \xi, \eta)$  is normal if and only if  $N_\varphi(X, Y) - d\eta(X, Y) \cdot \xi = 0$ . Moreover, in this case, the relations  $L_\xi \varphi = 0$  and  $(L_{\varphi X} \eta)Y - (L_{\varphi Y} \eta)X = 0$  are also implied. Indeed, taking  $Y := \xi$  in the previous relation we obtain

$$[X, \xi] - \varphi[\varphi X, \xi] + \xi(\eta(X)) \cdot \xi = 0,$$

for any  $X, Y \in \Gamma(TM)$  and for  $X \mapsto \varphi X$ , we get

$$0 = [\varphi X, \xi] - \varphi[\varphi^2 X, \xi] = -[\xi, \varphi X] + \varphi[\xi, X] = -(L_\xi \varphi)X.$$

But,  $(\varphi, \xi, \eta)$  is normal if the associated almost product structures  $E_1 := \varphi - \eta \otimes \xi$  and  $E_2 := \varphi + \eta \otimes \xi$  are integrable (that is, their Nijenhuis tensor fields vanish identically) and  $L_\xi \eta = 0$ . Applying  $\eta$  to  $N_{E_1}(\varphi X, Y) = 0$ , we obtain

$$(L_{\varphi^2 X} \eta)Y - (L_{\varphi Y} \eta)(\varphi X) = 0,$$

for any  $X, Y \in \Gamma(TM)$ , which is equivalent to

$$(L_X \eta)Y - (L_{\eta(X) \cdot \xi} \eta)Y - (L_{\varphi Y} \eta)(\varphi X) = 0,$$

for any  $X, Y \in \Gamma(TM)$ . For  $X \mapsto \varphi X$ , we get

$$(L_{\varphi X} \eta)Y - (L_{\varphi Y} \eta)X + (L_{\varphi Y} \eta)(\eta(X) \cdot \xi) = 0,$$

for any  $X, Y \in \Gamma(TM)$  and the last term is zero because

$$\begin{aligned} (L_{\varphi Y} \eta)(\eta(X) \cdot \xi) &:= (\varphi Y)(\eta(X)) - \eta([\varphi Y, \eta(X) \cdot \xi]) \\ &= (\varphi Y)(\eta(X)) + \eta(X)\eta([\xi, \varphi Y]) - (\varphi Y)(\eta(X)) \\ &= \eta(X)[\xi(\eta(\varphi Y)) - (\varphi Y)(\eta(\xi)) - (d\eta)(\xi, \varphi Y)] = 0. \end{aligned}$$

□

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